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New order parameters in the Potts model on a Cayley tree

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Abstract

For the q-state Potts model new order parameters projecting on a group of spins instead of a single spin are introduced. On a Cayley tree this allows the physical interpretation of the Potts model at noninteger values q_0 of the number of states. The model can be solved recursively. This recursion exhibits chaotic behaviour changing qualitatively at critical values of q_0. Using an additional order parameter belonging to a group of zero extrapolated size the additional ordering is related to a percolation problem. This percolation distinguishes different phases and explains the critical indices of percolation class occuring at the Peierls temperature.

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1. Introduction

An interesting feature of the q-state Potts model [1, 2] is the possibility of considering its thermodynamic observables as analytic functions of the number q of spin states. Extrapolating q to unphysical values ($q = 0$ [3] or $q = 1/2$ [4]) or taking the derivative at $q = 1$ [5] the observables of other models (resistor network, or dilute spin glass or percolation, respectively) are obtained. Since this equivalence holds only for the thermodynamic observables the connection to the dynamics of the original Potts model is lost. Extrapolation requires explicit expressions of the observables as functions of q. This can be achieved, if we consider the q-state Potts model on a Cayley tree, which can be solved with a recursion formalism [6–14]. Extrapolation in q can be done in the recursion for the order parameter to obtain relations to new models. As an example we mention the result [14] that the percolation case $q = 1$ is equivalent to the logistic equation, which describes the formation of percolation clusters. To interpret non-integer values of q we introduce a new order parameter. Instead of distinguishing one of the q values of the spin we assume ordering for a whole group of q_1 spins. A next-neighbour interaction favouring equal spins will also favour spins in the same group. Therefore it is not surprising that we find the same mean-field equations as in the case $q_1 = 1$,

however with an effective number of spins $q_0 = q/q_1$ and a redefined coupling $K(K_0)$ as a function of the original next-neighbour coupling K_0. Since q and q_1 may be chosen large, this allows the interpretation of the q-state Potts model with noninteger q_0 in terms of a physical model. Due to the redefinition of the coupling not all values of K may correspond to physical values of the original coupling K_0. However, an extrapolation in the coupling is much easier to perform than one in the number of components. An example will be given. At $q_0 = 1$ we encounter the logistic equation exhibiting chaotic behaviour depending on the value of K as shown in [14]. Using the conventional order parameter chaotic behaviour requires a next-nearest-neighbour interaction for integer $q > 1$ [15]. Since it can be observed for noninteger q and an external field [16], we expect chaotic behaviour to persist for $q_0 > 1$. One of the aims of the present paper is to study the transition into chaos for real values of q_0.

A problem on a Cayley tree occurs if the recursion allows for more than one stable solution. On a normal lattice surface effects can be neglected and the homogeneous system can be characterized by a free energy per site $f(m)$ as a function of the magnetization. The fixed-point solutions of the recursion correspond to different minima of f. The phase with the lowest value of f will be stable; the others are thermodynamically metastable. On a Cayley tree f cannot be obtained without further assumptions due to the presence of the border points. At a fixed point the interior of a large Cayley tree exhibits the same magnetization for all sites. As first treated in [8,17] for $q = 2$ and later on generalized for $q \geqslant 1$ by Peruggi [9] a free energy per site f can be found by integration. This f describes the interior of a large Cayley tree and can be taken as representative for the infinite Bethe lattice. At zero magnetic field one obtains [9] from the absolute minimum of f for ferromagnetic coupling one transition from the disordered state into a magnetized state and one transition for negative coupling into an antiferromagnetic state. The transition obtained by the presumably exact Bethe–Peierls approximation [18] corresponds to a spinodal point of a metastable phase. The picture of a homogeneous free energy seems not to be very natural on a Cayley tree since most of the sites are either boundary points or lie in the transition region. As shown in [11] and [14] the values of the surface magnetizations decide which of the stable fixed points is adopted. In this picture the phase previously discarded as metastable appearing at the Bethe–Peierls temperature corresponds to a physical phase. To reconcile the recursion on a Cayley tree with the approach of a homogeneous f various proposals have been made. The trivial solution [14] by chosing the boundary conditions such that f holds also for the Cayley tree does not look very natural. In [13] f ought to be computed by the difference of free energies between the tree and a slightly modified lattice, which is justified only if the solution is homogeneous. Monroe [19] conjectured that the most stable fixed point corresponds to the physical phase. This criterion reproduces at zero field the phase diagram of Peruggi [9]. The merit of this criterion is that it applies locally and does not rely on values at the border. In the present paper we adopt the picture of Aguiar *et al* [11] and our previous paper [14], that any stable fixed point leads to a physical phase. Within this picture two questions arise. The first question is how different phases can be distinguished by properties other than their behaviour at the border. The second question is related to the end points of the disordered phase. These are second-order phase transitions with critical indices belonging to the percolation class [14, 20]. To explain this somewhat surprising behaviour we introduce besides a normal ordering parameter with respect to one value of the spins a second ordering on an additional group. By extrapolating the size of the latter to zero we shall see that this ordering describes a bond percolation problem with a probability depending on the first-order parameter. At the end points of the disordered phase also the percolation becomes critical, which explains the occurrence of percolation indices. If there are different stable fixed points possible, they will lead to different bond probabilities. Therefore the corresponding phases can be distinguished apart from the boundary conditions by their different local percolation properties.

The paper is organized in the following way. In section 2 we give the general formulae, especially for the the new order parameter. The mean-field equation and possible chaotic solutions are discussed in section 3. In section 4 we present a q-state Potts model with an additional long-range force allowing a physical interpretation of the coupling K. Section 5 is devoted to the appearance of percolation transitions in the q-state Potts model and section 6 contains our conclusions.

2. General formalism

We consider the q-state Potts model on a Cayley tree with coordination number z. At each site x exists a spin σ_x taking q different values (i.e. $\sigma_x = 1, 2, \ldots, q$). The spins interact with their next neighbours along the bonds $\langle x, y\rangle$ described by the following Hamiltonian:

$$-\beta H = K_0 \sum_{\langle x,y\rangle} \delta_{\sigma_x,\sigma_y}. \tag{1}$$

K_0 is proportional to the inverse temperature and the interaction strength. Negative K_0 values correspond to antiferromagnetic coupling. The partition function and Boltzmann distribution of a single spin can be calculated recursively with the methods described in [14]. A Cayley tree can be thought of as z branches of length n connected at a site with spin σ. Since the Hamiltonian (1) can be written as a sum over the branches, we consider the partition function $T_n(\sigma)$ for a branch where all spin summations inside the branch except the first are carried out. Any ordering appears as a nontrivial σ dependence of $T_n(\sigma)$. For the description of this ordering we decompose the spin values into $P+1$ groups with size $q_i (i = 1, 2, \ldots, P+1)$. Each group i is characterized by a projector Δ^i with the property

$$\Delta^i_\sigma = \begin{cases} 1 & \sigma \text{ in group } i \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The group ordering of $T_n(\sigma)$ can be written as

$$T_n(\sigma) = \sum_{i=1}^{P+1} u_{in}\ \Delta^i_\sigma. \tag{3}$$

The disordered state corresponds to value of $u_{in} = u_n$ independent of i. The conventional ordering with respect to a single value $\bar\sigma$ is obtained with $P = 1$ and $\Delta^1_\sigma = \delta_{\sigma,\bar\sigma}$. This ordering is for the Ising case $q = 2$ also the most general one. For the parameters u_{in} one can find a coupled system of recursion relations in the following way. Each branch $T_n(\sigma)$ can be related according to figure 1 to $z-1$ branches of length $n-1$ by

$$T_n(\sigma) = \sum_{\sigma'} \exp\left(K_0 \delta_{\sigma,\sigma'}\right) \left(T_{n-1}(\sigma')\right)^{z-1}. \tag{4}$$

Parametrizing $T_n(\sigma)$ as in equation (3) the products in (4) can be evaluated using the relation

$$\Delta^i_\sigma \Delta^j_\sigma = \delta_{ij} \Delta^i_\sigma. \tag{5}$$

Comparing coefficients on the two sides we find

$$u_{i,n} = (\mathrm{e}^{K_0} - 1)(u_{i,n-1})^{z-1} + \sum_j q_j (u_{j,n-1})^{z-1}. \tag{6}$$

Spin expectation values can depend only on the ratios of $u_{i,n}$. For these we obtain the recursion

$$\frac{u_{i,n}}{u_{1,n}} = 1 + \frac{(\mathrm{e}^{K_0} - 1)((u_{i,n-1}/u_{1,n-1})^{z-1} - 1)}{\mathrm{e}^{K_0} - 1 + \sum_j q_j (u_{j,n-1}/u_{1,n-1})^{z-1}}. \tag{7}$$

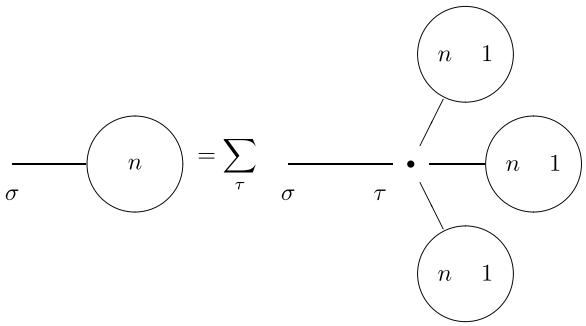

Figure 1. Recursion for $z = 4$.

The Boltzmann weight for the central spin at a fixed point u_i/u_1 is given by

$$w(\sigma) = N\,(T(\sigma))^z \tag{8}$$

where the normalization constant follows from $\sum w(\sigma) = 1$. With the distribution (8) we obtain for $\langle \Delta^i \rangle$

$$\langle \Delta^i \rangle = \frac{q_i (u_i/u_1)^z}{q_1 + \sum_{j>1} q_j (u_j/u_1)^z}. \tag{9}$$

The spin expectation values $\langle \Delta^i \rangle$ lead to the magnetizations of a site. For these we use the linear combinations

$$m_i = \frac{1}{q - q_i} \langle q\,\Delta^i - q_i \rangle \tag{10}$$

$$\overline{m}_i = 1 - \frac{q}{q_i} \langle \Delta^i \rangle. \tag{11}$$

The coefficients have been chosen such that $m_i = \overline{m}_i = 0$ holds for a disordered state $\langle \Delta^i \rangle = q_i/q$ and $m_i = 1$ ($\overline{m}_i = 1$) for the ordered case where all (none) of the spin values σ belong to group i. Among these $2(P+2)$ magnetizations only P are linearly independent.

We are mainly interested in the properties of only one ordering ($P = 1$) with respect to a group of size $q = q_1$. As order parameter we use $x_n = 1 - u_{2n}/u_{1n}$. Equation (7) reduces to a recursion formula for x

$$x_{n+1} = p(x_n)(1 - (1 - x_n)^{z-1}) \tag{12}$$

with

$$p(x) = (\mathrm{e}^{K_0} - 1)/(\mathrm{e}^{K_0} - 1 + q_1 + (q - q_1)(1 - x)^{z-1}). \tag{13}$$

For $P = 2$ we treat only the case where the recursion system decouples. This happens for the extrapolation $q_3 \to 0$. In addition to the recursion (12) we obtain a quadratic recursion for $y_n = 1 - u_{3n}/u_{1n}$

$$y_{n+1} = p(x_n)(1 - (1 - y_n)^{z-1}). \tag{14}$$

For fixed points $x_n = x$ the logistic equation (14) describes a bond percolation problem with probability $p(x)$ [14]. We use as independent magnetizations m_1 and $\overline{m}_3$ given by

$$m_1 = q_1 \left(1 - (1 - x)^z\right)/N \tag{15}$$

and

$$\overline{m}_3 = 1 - q(1 - y)^z/N \tag{16}$$

with the normalization factor $N = q_1 + (q - q_1)(1 - x)^z$. Even the order parameter y no longer appears in T (since $\Delta_\sigma^3 \to 0$ for $q_3 \to 0$); the extrapolated value of $\overline{m}_3$ remains finite and can be related to the percolation ordering parameter m_P defined by

$$m_P = 1 - (1 - y)^z \tag{17}$$

through the relation

$$\overline{m}_3 = m_P - (q/q_1 - 1)m_1(1 - m_P). \tag{18}$$

Equation (12) will be discussed in detail in the next section. The relation of percolation to the phase transitions of the q-state Potts model given by (14) will be treated in section 5.

3. Potts model at noninteger values of q

One of the interesting properties of (12) is the observation that it depends only on two combinations of the three parameters K_0, q and q_1. If we redefine the inverse temperature K by

$$e^K - 1 = \frac{1}{q_1}(e^{K_0} - 1) \tag{19}$$

and introduce an effective number of spin states

$$q_0 = q/q_1 \tag{20}$$

equation (12) can be written as a mapping

$$x_{n+1} = f(x_n) \tag{21}$$

with

$$f(x) = \frac{(e^K - 1)(1 - (1 - x)^{z-1})}{e^K + (q_0 - 1)(1 - x)^{z-1}}. \tag{22}$$

The recursion (21) is identical to a problem with a standard ordering parameter $q_1 = 1$ at temperature K with the difference that the number of states q_0 can approximate any real number $q_0 \geqslant 1$ for sufficiently large $q_1 \leqslant q$. Therefore the extrapolation of the q-state Potts model to noninteger values of q_0 is equivalent to group ordering. In the range $e^K - 1 \geqslant q_0/z$ the only stable orbits of the recursion (21) are fixed points. These correspond to various phases to be discussed in section 5. In the remainder of this section we discuss (21) for real negative K and odd z. For simplicity we take $z = 3$. For $q_0 = 1$ the mapping (21) corresponds to the logistic equation, which exhibits deterministic chaos for the control parameter e^K for the range $1/2 \leqslant e^K \leqslant 2/3$. For small deviations of q_0 from 1 we expect the mapping (21) to have a similar behaviour. In figure 2 we show the iterates of (21) for large n as a function of K at $q_0 = 1.08$. This bifurcation diagram is similar to that of the logistic equation [21]. In the chaotic regions any x_n is possible within the bounds

$$f(1 - e^{-K}) \leqslant (1 - x) \leqslant e^{-K}. \tag{23}$$

However, there exists a characteristic difference to the logistic equation. For the latter all iterates tend to ∞ for $e^{-K} \geqslant 2$, whereas for (21) the chaotic regime ends. This can be explained by the existence of a pair of fixed points of (21):

$$x_\pm = 1 + \frac{1}{2(q_0 - 1)}\left(1 - e^K \pm \sqrt{(1 - e^K)^2 - 4(q_0 - 1)}\right) \tag{24}$$

possible in the range

$$|e^K - 1| \geqslant 2\sqrt{q_0 - 1}. \tag{25}$$

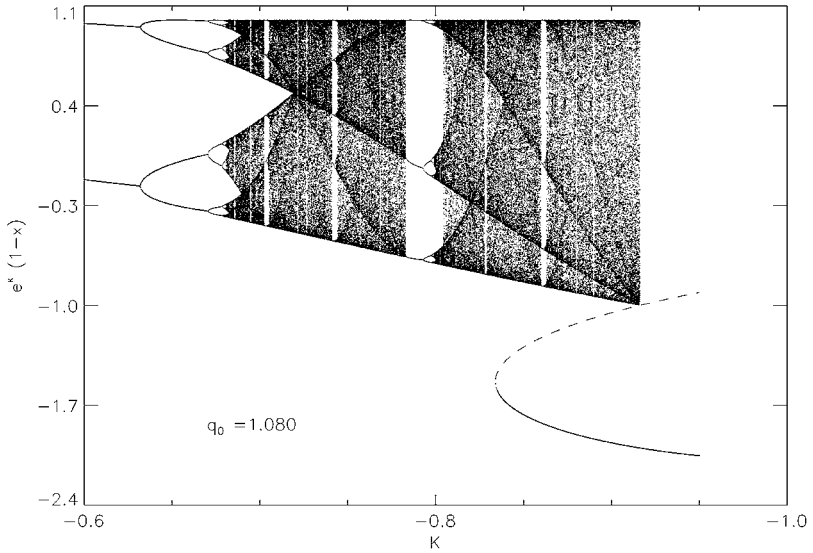

Figure 2. Iterates of the recursion (21) as a function of K at $q_0 = 1.08$. The curves give the fixed points from equation (24).

x_+ is stable (solid curve in figure 2) and x_- (broken curve) is unstable for $K \leqslant 0$. The basin of attraction of x_+ is given by initial values $x_0 \geqslant x_-$. Therefore the chaotic regime ends where the lower bound in (23) is equal to x_- from (24). Solving this equation for K we find the endpoint K_E of the chaotic regime

$$\mathrm{e}^{K_\mathrm{E}} = \tfrac{1}{4}\left(1 + \sqrt{9 - 8q_0}\right). \tag{26}$$

If we increase q beyond $10/9$ the endpoint of the chaotic regime is determined by equality in (25). At the same time the pattern of the chaotic behaviour changes qualitatively. An example for $q_0 = 1.2$ is shown in figure 3. The regions of chaos are intercepted by stable orbits with increasing periods of $\tau = 2, 3, \ldots$. If we enlarge the region in between the stable orbits they are similar to the pattern seen in figure 2. The stable orbits exhibit an accumulation point at the end of the chaotic region given by the equality in (25). The period τ of the orbit as a function of K may be estimated (for sufficiently large τ) by replacing the recursion formula by a differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}n} \approx x_{n+1} - x_n = f(x) - x. \tag{27}$$

The solutions $x(n)$ of (27) can be found by an elementary integration. They are periodic with a period

$$\tau = \frac{2\pi(1 - \mathrm{e}^K)(2 - q_0)}{(1 - \mathrm{e}^K + q_0) + \sqrt{4(q_0 - 1) - (\mathrm{e}^K - 1)^2}}. \tag{28}$$

In figure 4 we compare this $\tau(K)$ with the integer length of the orbits in figure 3. As corresponding K value we take the first bifurcation in each interval. As one sees from figure 4, the agreement between the observed periods and the estimate (28) is excellent, even at small τ. For values of q_0 larger than $5/4$ the fixed points (24) no longer exist. The periods as seen in figures 5 ($q_0 = 1.255$) and 6 ($q_0 = 1.275$) remain finite. This is confirmed by the

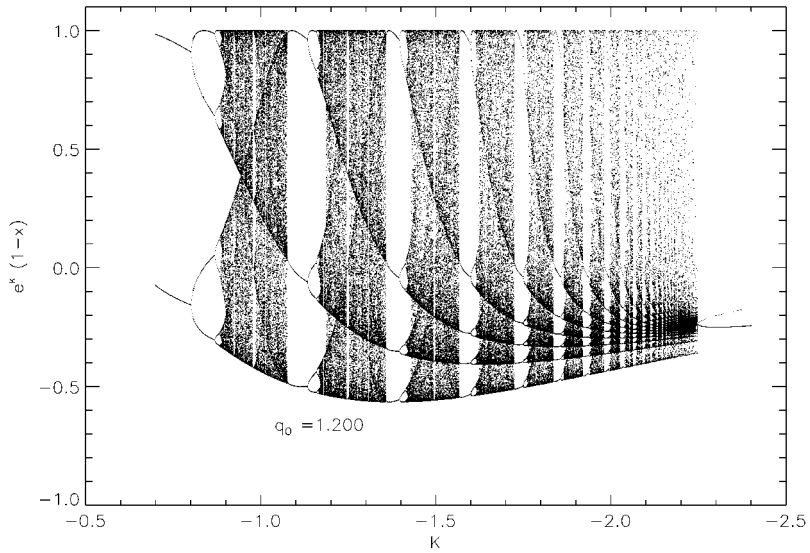

Figure 3. Iterates of the recursion (21) as a function of K at $q_0 = 1.20$. Chaotic regions are intercepted by stable orbits with increasing periods. The accumulation point of these orbits is given by equation (25).

Table 1. Characteristics of chaotic behaviour in different intervals of q_0. The value of q_c, above which no orbit of infinite length occurs, is given numerically by $q_c = 1.53$.

q range	End point $\exp(K)$ of chaotic behaviour	Repetition of deterministic chaos	Maximum orbit length
$1 \leqslant q_0 \leqslant 10/9$	$\frac{1}{4}\left(1+\sqrt{9-8q_0}\right)$	1	∞
$10/9 < q_0 \leqslant 5/4$	$1-2\sqrt{q_0-1}$	∞	∞
$5/4 < q_0 < q_c$	0	$2\pi(2-q_0)/(q_0+1)\sqrt{4q_0-5}$	∞
$q_0 > q_c$	—	0	finite
$1.56 < q_0 < 3$	—	0	2

agreement with (28) in figure 4. The last chaotic region extends to $K \to -\infty$ and the number of regions of stable orbits decreases with increasing q_0. For $q_0 > q_c$ with $q_c \approx 1.53$ there will be no transition into chaos. For $1.56 \leqslant q_0 \leqslant 3$ only a $\tau = 2$ orbit exists. For $q_0 \geqslant 3$ and negative coupling K we have only the disordered fixed point $x = 0$. Examples of the behaviour seen in figures 2 and 6 without detailed discussion have been given in [16].

We see that both the recursion (21) for general q_0 and the logistic equation for $q_0 = 1$ have a control parameter K which decides on the details of the chaotic behaviour. In addition the parameter $q_0 = q/q_1$ decides which type of chaotic behaviour occurs. The various types we found are summarized in table 1.

4. A generalized Potts model

As shown in the previous section a Potts model extrapolated to noninteger values q_0 is equivalent to a q-state Potts model at a rescaled temperature using an order parameter describing simultaneous ordering of a group of q_1 spins. The restriction to physical values of K_0 imposes by (19) a lower limit of K, which excludes the chaotic behaviour discussed in section 3.

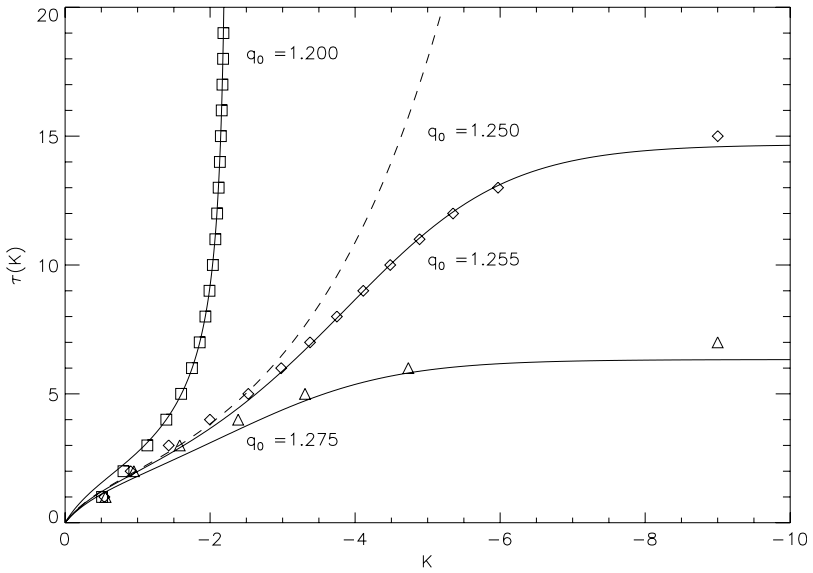

Figure 4. The continuum period (28) as a function of K for various values of q_0. The data points are estimates from figures 3, 5 and 6 using as K value the locus of the first period doubling. The dashed curve gives the critical curve for $q_0 = 5/4$.

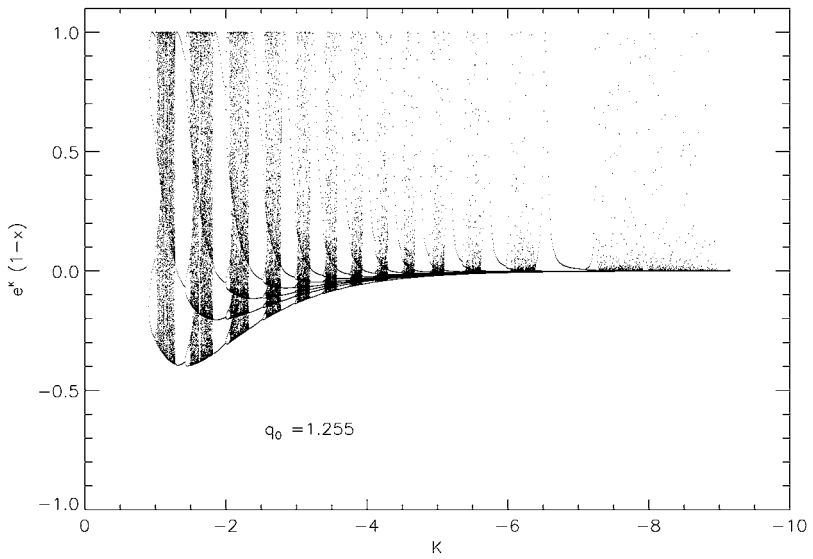

Figure 5. Iterates of the recursion (21) as a function of K at $q_0 = 1.255$. Different chaotic regimes are intercepted by a finite number of stable orbits.

Therefore an extrapolation is still required. To find a model where K has a physical meaning, we introduce the number E of occupied links in each spin configuration:

$$E = \sum_{\langle x,y \rangle} \delta_{\sigma_x,\sigma_y}. \tag{29}$$

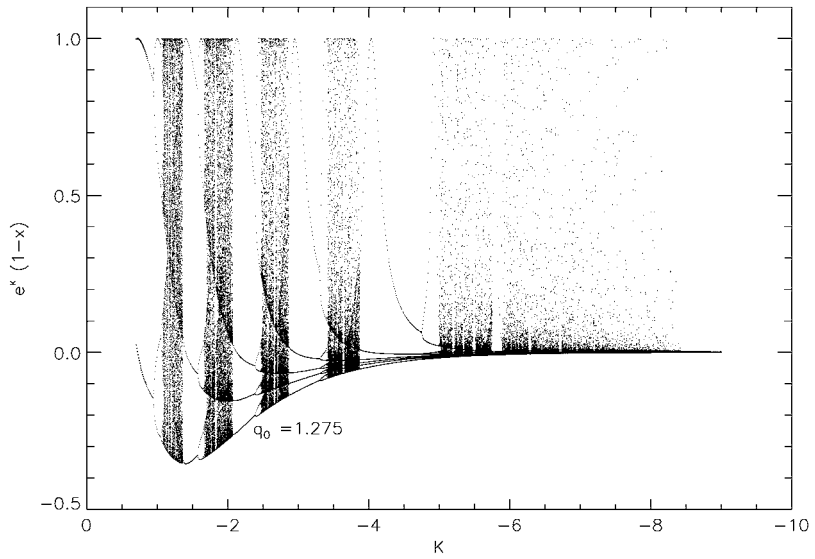

Figure 6. Iterates of the recursion (21) as a function of K at $q_0 = 1.275$. Different chaotic regimes are intercepted by a finite number of stable orbits.

A new model may be defined by allowing only spin configurations where E is even or odd. This corresponds to a nonlocal spin interaction. The partitioned sum for even E then will involve branch contributions with both even and odd E. It is convenient to define partitioned sums $T_n^{\mathrm{e}}(\sigma)$ or $T_n^{\mathrm{o}}(\sigma)$ for branches allowing only even or odd values for E in the branch. As in the previous case we express $T_n^{\mathrm{e,o}}(\sigma)$ by contributions of the subbranches $T_{n-1}^{\mathrm{e,o}}(\sigma)$. This system decouples, if we consider the sum and difference of even and odd branches given by

$$T_n^{(\pm)}(\sigma) = T_n^{\mathrm{e}}(\sigma) \pm T_n^{\mathrm{o}}(\sigma). \tag{30}$$

If we parametrize $T_n^{(\pm)}$ in the form (3)

$$T_n^{(\pm)}(\sigma) = \sum_i^{P+1} u_{i,n}^{(\pm)} \Delta_\sigma^i \tag{31}$$

$T_n^{(+)}$ corresponds to the unrestricted case given by (3) and (6). Using only one order parameter ($P = 1$) and introducing the ratio

$$x_n^{(-)} = \frac{u_{2,n}^{(-)}}{u_{1,n}^{(-)}} - 1 \tag{32}$$

we obtain the recursion formula

$$x_{n+1}^{(-)} = \frac{(1+\mathrm{e}^{K_0})((1-x_n^{(-)})^{z-1}-1)}{q_1 - 1 - \mathrm{e}^{K_0} + (q-q_1)(1-x_n^{(-)})^{z-1}}. \tag{33}$$

Redefining the temperature by

$$\mathrm{e}^{K_0} + 1 = q_1(1-\mathrm{e}^{K}) \tag{34}$$

we obtain with $q_0 = q/q_1$ the same recursion formula as (21) without restriction on K. The chaotic region of (21) can be reached for large q_1 and fixed value for K_0. In this range the according recursion to $x_n^{(+)}$ leads always to the trivial stable fixed point $x^{(+)} = 0$.

5. Phases of the Potts model

The stable fixed points of recursion (12) give in our interpretation the possible phases with respect to the group ordering parameter x. We do not perform the replacements (19) and (20), but describe the model in terms of the original parameters K_0, q and q_1. There exist three critical points K_c, K_c' and K_c''. The disordered fixed point $x = 0$ is stable at high temperatures in the range $K_c' < K_0 < K_c$ with $K_c' = \ln(1 - q/z)$ and $K_c = \ln(1 + q/(z-2))$. As the temperature is lowered two new fixed points given by (24) appear for $K_0 > K_c''$. Only x_+ with positive magnetization m_1 corresponds to a stable fixed point. For $z = 3$ the value of x_+ is given by equation (24). In contrast to K_0 and K_0' the value of K_c'' depends on q_1. For $z = 3$ one finds $K_c'' = \ln(1 + 2\sqrt{q_1(q - q_1)})$. Therefore there exists a series of critical points depending on q_1. For $K_0 > K_c$ the disordered phase changes into the fixed point x_- (for $z = 3$ given by equation (24)) with negative magnetization m_1, and for $K_0 < K_c'$ we encounter an antiferromagnetic coupling. This transition corresponds to the first bifurcation discussed in section 3. If more than one stable fixed point exists, the value at the boundary decides the adopted phase. The transition at K_c'' is of first order. For the following it does not matter whether this transition appears at K_c'' or a higher value as advocated in [9] from equality of the free energies. It is only important that the transitions at K_c and K_c' are genuine transitions and do not correspond to spinodal points. For the critical indices at K_c we find values belonging to the percolation class ($\beta = \gamma = 1$). The same is true for the average magnetization also at K_c'. The staggered magnetization equal to the difference of magnetizations at adjacent sites exhibits at K_c' indices of the mean-field class ($2\beta = \gamma = 1$). The Ising case $q_1 = q/2$ is exceptional. Since the amplitude of the average magnetization vanishes due to the Ising symmetry on an AB lattice ($m(x_+) + m(x_-) = 0$) the percolation transitions dissappear, and one observes mean-field indices at both critical points K_c' and $K_c'' = K_c$. Note that a Ising-type behaviour occurs not only for $q = 2q_1 = 2$ but also for any even q with the choice $q_1 = q/2$. This reflects the fact that the recursion (12) can be written in the form (21), which depends only on the ratio $q_0 = q/q_1$.

The appearance of percolation indices for $q_1 \neq q/2$ is somewhat surprising. To understand the mechanism we consider the recursion (12) for the order parameter x together with an extrapolated second-order parameter y_n given by (14). The recursion for the latter reads at a fixed point for x as

$$y_{n+1} = p(x)(1 - (1 - y_n)^{z-1}) \tag{35}$$

with $p(x)$ given by (13). As shown in [14], the recursion (35) is equivalent to bond percolation with a probability $|p(x)|$. A nonzero fixed-point value y leads to a finite probability m_P for a site belonging to the infinite cluster

$$m_P = 1 - (1 - y)^z. \tag{36}$$

The mapping (35) has the fixed points $y = 0$ and x. The stable fixed point must satisfy

$$|p(x)(z-1)(1-y)^{z-2}| < 1. \tag{37}$$

Both the disordered fixed point $x = 0$ in the range $K_c' < K_0 < K_c$ and x_- with negative magnetization for $K_0 > K_c$ lead to the nonpercolating state $y = 0$. In contrast the fixed point x_+ requires $y = x_+$. Therefore the two phases possible for $K_0 > K_c''$ can be distinguished locally by their percolating properties. The phase with $m_1 > 0$ exhibits an infinite percolation cluster, whereas the phase with $m_1 \leqslant 0$ is in the disordered percolation state. Using the criterion of existence of an infinite cluster the phase on the Cayley tree is uniquely defined and can be carried over to the infinite Bethe lattice, since one no longer needs the values at the boundary. For $K_0 < K_c'$ the recursion (12) exhibits a $\tau = 2$ orbit corresponding to antiferromagnetic ordering.

By a straightforward calculation one can show that the recursion (14) has a stable fixed point $y = 0$, although p alternates between $p(x_\pm)$ in subsequent iterations.

If K_0 approaches the end points K_c or K_c' of the disordered phase, $|p(x)|$ approaches the critical value $1/(z-1)$. At these points all moments of the cluster size distribution diverge. These moments can be represented by derivatives of the magnetization m_P with respect to an external field. Therefore magnetization and susceptibility will be singular with critical indices belonging to the percolation class. Since the various magnetizations are not independent due to relation (18), these divergences will also show up in magnetizations other than m_P.

6. Conclusions

We introduced a new type of ordering in the q-state Potts model where a spin belongs to one or more groups of values instead of the conventional ordering with respect to one value. The recursion formalism for the latter can be also applied for the new ordering. For only one group of size q_1 the recursions can be cast into the same form as for $q_1 = 1$ with the exception that the effective number of states is given by $q_0 = q/q_1$, which can approximate any real number larger than unity. This allows an extrapolation of the q-state Potts model to noninteger values of q without losing the physical interpretation. For q_0 close to unity the recursion exhibits deterministic chaos similar to the logistic equation. The control parameter is essentially the temperature K^{-1}. Increasing q_0 we observe as a function of K repetitions of deterministic chaos separated by stable orbits of length $\tau = 2, 3, 4, \ldots$. The number of orbits and their length can be estimated by a continuous length approximation. Most of this chaotic behaviour occurs at $K(K_0)$ values which cannot be reached with real values of K_0. To avoid this extrapolation to unphysical values of the temperature one can introduce a generalized Potts model, where possible spin configurations are restricted by the condition that the number of links connecting sites with equal spin values must be even or odd. In [10] the Potts model has been extrapolated to real values of q using the analytic expression for the free energy. For $q < 2$ negative values for the specific heat have been found. This unphysical behaviour can be avoided if one uses the new order parameter referring to a group of q_1 spins and interprets the ratio $q_0 = q/q_1$ as the effective number of spins.

The phase structure of the q-state Potts model with ordering with respect to one group of size q_1 is qualitatively similar to the conventional $q_1 = 1$ case. Only for even q does the Ising case $q_1 = q/2$ behave differently. For $q_1 \neq q/2$ the transitions at the endpoints of the disordered phase are of second order with percolation class critical indices. To understand their occurrence we added a second group of ordering with size extrapolated to zero. This does not change the recursions for the first group. In addition one obtains a logistic equation for the second order parameter, which leads to indices of the percolation class. Since the control parameter in the logistic equation depends on the phase, different phases can be distinguished by their percolation property. The phases with positive magnetization exhibit an infinite cluster with finite probability, whereas negative or zero magnetizations $\leqslant 0$ have zero probability. This criterion allows a local distinction between two possible phases. Since one does not have to resort to boundary conditions, this criterion can also be applied on the infinite Bethe lattice.

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